Formalisation of CW complexes

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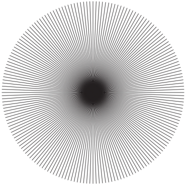
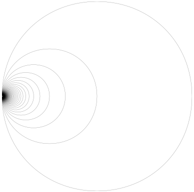
Why CW complexes?

- Very general class of spaces
  - Examples of CW complexes: $\mathbb{R}^n$, S^n, $\mathbb{CP}^n$, $\mathbb{RP}^\infty$
  - Homotopy type of CW complexes: differentiable manifolds
  - Not a CW complex: hedgehog space
  - Not homotopy equivalent to a CW complex: Hawaiian earring



hedgehog space



Hawaiian earring

Why CW complexes?

- A lot of strong results about CW complexes

Theorem (Whitehead theorem, 1949)

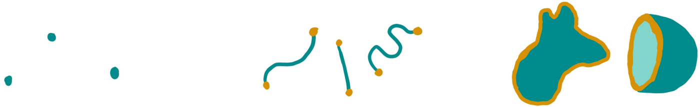
A continuous map between two CW complexes that induces isomorphisms on all homotopy groups is a homotopy equivalence.

Theorem (Cellular homology)

Let X be a CW complex. Then the cellular and singular homology of X agree.

Intuition: What is a CW complex?

- Glue *n*-cells (i.e. continuous images of *n*-discs) together along their boundaries

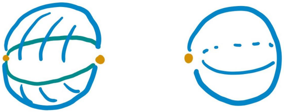


0-cells 1-cells 2-cells

Examples: What is a CW complex?

Interval

Real line



2-sphere

Definition: What is a CW complex?

Let X be a Hausdorff space. An *(absolute) CW complex* on X consists of a family of indexing sets $(I_n)_{n\in\mathbb{N}}$ and a family of continuous maps $(Q_i^n\colon D^n \to X)_{n\in\mathbb{N}, i\in I_n}$ called *characteristic maps* with the following properties:

(i) $Q_i^n|_{\mathrm{int}(D^n)} : \mathrm{int}(D^n) \to Q_i^n(\mathrm{int}(D^n))$ is a homeomorphism for every $n \in \mathbb{N}$ and $i \in I_n$. We call $e_i^n := Q_i^n(\mathrm{int}(D^n))$ an *(open) n-cell* and $\overline{e}_i^n := Q_i^n(D^n)$ a *closed n-cell*.

(ii) Two different open cells are disjoint.

(iii) For each $n \in \mathbb{N}$ and $i \in I_n$ the *cell frontier* $\partial e_i^n := Q_i^n(\partial D^n)$ is contained in the union of a finite number of closed cells of a lower dimension.

(iv) A set $A \subseteq X$ is closed if the intersections $A \cap \overline{e}_i^n$ are closed for all $n \in \mathbb{N}$ and $i \in I_n$.

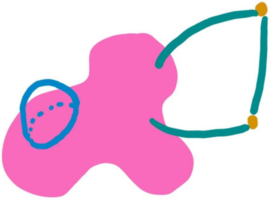
(v) The union of all closed cells is X.

Lean: What is a CW complex?

```
class CWComplex.{u} {X : Type u} [TopologicalSpace X] (C : Set X) where
  cell (n : ℕ) : Type u
  map (n : ℕ) (i : cell n) : PartialEquiv (Fin n → ℝ) X
  source_eq (n : ℕ) (i : cell n) : (map n i).source = ball 0 1
  continuousOn (n : ℕ) (i : cell n) : ContinuousOn (map n i) (closedBall 0 1)
  continuousOn_symm (n : ℕ) (i : cell n) : ContinuousOn (map n i).symm (map n i).target
  pairwiseDisjoint' :
    (univ : Set (Σ n, cell n)).PairwiseDisjoint (fun ni ↦ map ni.1 ni.2 '' ball 0 1)
  mapsTo' (n : ℕ) (i : cell n) : ∃ I : Π m, Finset (cell m),
    MapsTo (map n i) (sphere 0 1) (⋃ (m < n) (j ∈ I m), map m j '' closedBall 0 1)
  closed' (A : Set X) (hAC : A ⊆ C) :
    (∀ n j, IsClosed (A ∩ map n j '' closedBall 0 1)) → IsClosed A
  union' : ⋃ (n : ℕ) (j : cell n), map n j '' closedBall 0 1 = C
```

Intuition: What is a relative CW complex?

A relative CW complex additionally has a base set that the boundaries can attach to.



An example of a relative CW complex

Lean: What is a relative CW complex?

```
class RelCWComplex.{u} {X : Type u} [TopologicalSpace X] (C : Set X) (D : outParam (Set X)) where
  cell (n : ℕ) : Type u
  map (n : ℕ) (i : cell n) : PartialEquiv (Fin n → ℝ) X
  source_eq (n : ℕ) (i : cell n) : (map n i).source = ball 0 1
  continuousOn (n : ℕ) (i : cell n) : ContinuousOn (map n i) (closedBall 0 1)
  continuousOn_symm (n : ℕ) (i : cell n) : ContinuousOn (map n i).symm (map n i).target
  pairwiseDisjoint' :
    (univ : Set (Σ n, cell n)).PairwiseDisjoint (fun ni ↦ map ni.1 ni.2 '' ball 0 1)
  disjointBase' (n : ℕ) (i : cell n) : Disjoint (map n i '' ball 0 1) D
  mapsTo (n : ℕ) (i : cell n) : ∃ I : Π m, Finset (cell m),
    MapsTo (map n i) (sphere 0 1) (D ∪ ⋃ (m < n) (j ∈ I m), map m j '' closedBall 0 1)
  closed' (A : Set X) (hAC : A ⊆ C) :
    ((∀ n j, IsClosed (A ∩ map n j '' closedBall 0 1)) ∧ IsClosed (A ∩ D)) → IsClosed A
  isClosedBase : IsClosed D
  union' : D ∪ ⋃ (n : ℕ) (j : cell n), map n j '' closedBall 0 1 = C
```

Implementation: general situation

Situation: We have a general and a specific definition where

- the specific definition is a lot more commonly used
- the specific case provides significant simplifications
- the differentiating parameter is an `outParam`

Implementation: Issues with naive definition

- Naive approach: define an absolute CW complex as a relative one with empty base
- Issues with naive approach:
  - repeated simplifications
  - instances where the base is provably but not definitionally equal to empty set

Implementation: Issues with naive definition

- Product of two relative CW complexes $(C, \varnothing)$ and $(E, \varnothing)$ has type:

```
RelCWComplex (C ×ˢ E) (∅ ×ˢ E ∪ C ×ˢ ∅)
```

- Product of two absolute CW complexes C and E has type:

```
CWComplex (C ×ˢ E)
```

- With the naive approach this would be definitionally the same as:

```
RelCWComplex (C ×ˢ E) ∅
```

What has been done in Lean?

By other people (that I am aware of):

- Categorical definition `TopCat.RelativeCWComplex` by Jiazhen Xia and Elliot Dean Young and refactored by Joël Riou: in Mathlib
- Whitehead theorem in model categories by Joël Riou: in Mathlib
- Equivalence of the definitions by Robert Maxton: PRs

What has been done in Lean?

By us:

- Definition and basic properties ($\sim$ 600 LOC): in Mathlib
- Finiteness notions ($\sim$ 300 LOC): in Mathlib
- Subcomplexes ($\sim$ 800 LOC): in Mathlib/PRs
- Compactly coherent spaces ($\sim$ 200 LOC): in Mathlib/PRs
- Product ($\sim$ 600 LOC): done
- Examples ($\sim$ 1000 LOC): needs refactor
- Rest of the Project ($\sim$ 3000 LOC)

Products of CW complexes

Let X and Y be CW complexes. The respective families of characteristic maps are $(Q_i^n \colon D^n \to X)_{n\in\mathbb{N}, i\in I_n}$ and $(P_j^m \colon D^m \to Y)_{m\in\mathbb{N}, j\in J_m}$.

Theorem

Assume that $X \times Y$ is compactly coherent. Then $X \times Y$ is a CW complex with characteristic maps $(Q_i^n \times P_j^m \colon D^n \times D^m \to C \times E)_{n,m\in\mathbb{N}, i\in I_n, j\in J_m}$ and indexing sets $K_l = \bigcup_{n+m=l} I_n \times J_m$.

Theorem

In general, the compact coherentification of $X \times Y$ is a CW complex.

Compactly coherent spaces

Abbreviation	Meaning summary
CG-1	Topology coherent with family of its compact subspaces
CG-2	Topology same as final topology with respect to continuous maps from arbitrary compact Hausdorff spaces
CG-3	Topology coherent with family of its compact Hausdorff subspaces

Definition

Let X be a topological space. We call X *compactly coherent* if a set $A \subseteq X$ is open iff for all compact sets $C \subseteq X$, the intersection $A \cap C$ is open in C.

Compactly coherent spaces

```
class CompactlyCoherentSpace (X : Type*) [TopologicalSpace X] : Prop where
  isCoherentWith : IsCoherentWith (X := X) {K | IsCompact K}
```

```
structure IsCoherentWith (S : Set (Set X)) : Prop where
  isOpen_of_forall_induced (u : Set X) :
    (∀ s ∈ S, IsOpen ((↑) ⁻¹' u : Set s)) → IsOpen u
```

Summary

- CW complexes are an important class of topological spaces
- a CW complex is made up of a lot of discs glued together
- the product of two CW complexes is in general **not** a CW complex
- some of the theory of CW complexes is already in mathlib!